Combining Noisy Measurements

Combining Noisy Measurements

- suppose that you take a measurement x_1 of some real-valued quantity (distance, velocity, etc.)
- your friend takes a second measurement x_2 of the same quantity
- after comparing the measurements you find that

$$x_1 \neq x_2$$

- what is the best estimate of the true value μ?

Combining Noisy Measurements

- suppose that an appropriate noise model for the measurements is

true, unknown value of x

$$x_1 = x + \varepsilon_{\sigma^2}$$

$$x_2 = x + \varepsilon_{\sigma^2}$$

where ε_{σ^2} is zero-mean Gaussian noise with variance σ^2

- because two different people are performing the measurements it might be reasonable to assume that x_1 and x_2 are independent

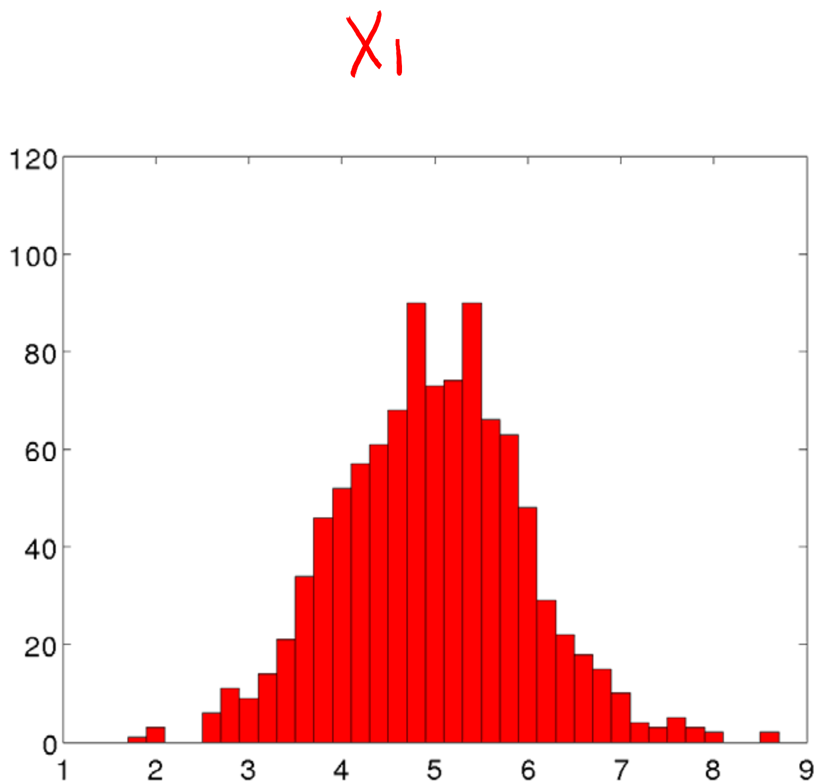
Combining Noisy Measurements

1000 noisy measurements of x
- zero-mean noise with variance = 1

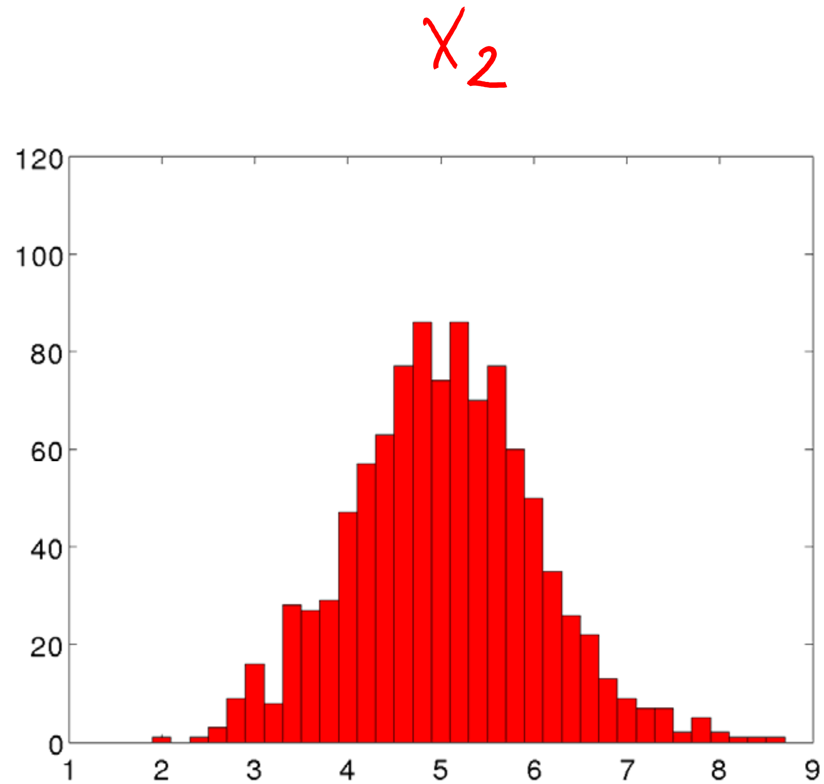
```
x = 5;
x1 = x + randn(1, 1000);   % noise variance = 1
x2 = x + randn(1, 1000);   % noise variance = 1
mu2 = (x1 + x2) / 2;

bins = 1:0.2:9;
hist(x1, bins);
hist(x2, bins);
hist(mu2, bins);
```

Combining Noisy Measurements



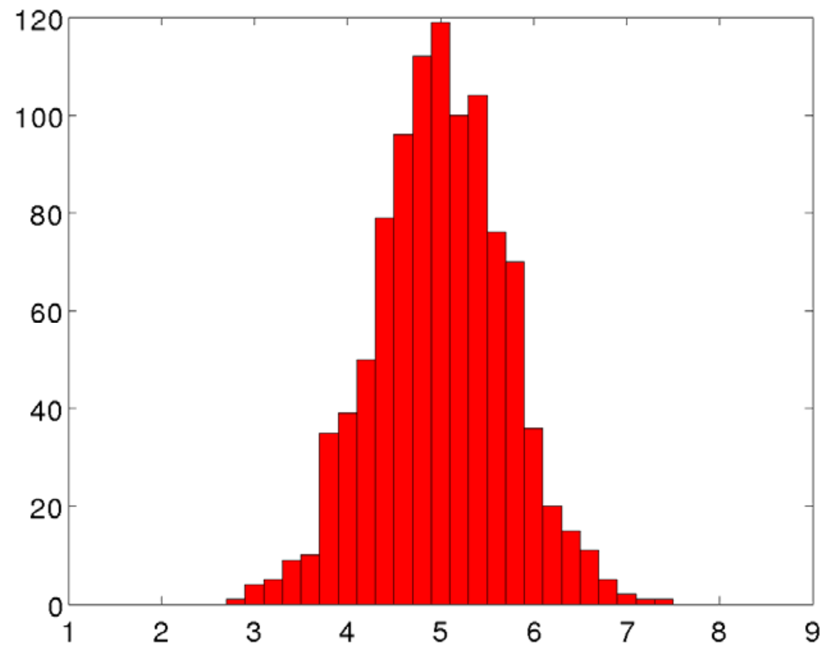
var(x1) = 0.9979



var(x2) = 0.9972

Combining Noisy Measurements

μ



var(mu2) = 0.4942

by averaging the values our estimate of x is more certain

Combining Noisy Measurements

- suppose the precision of your measurements is much worse than that of your friend
- consider the measurement noise model

$$x_1 = x + 3\varepsilon_{\sigma^2}$$

$$x_2 = x + \varepsilon_{\sigma^2}$$

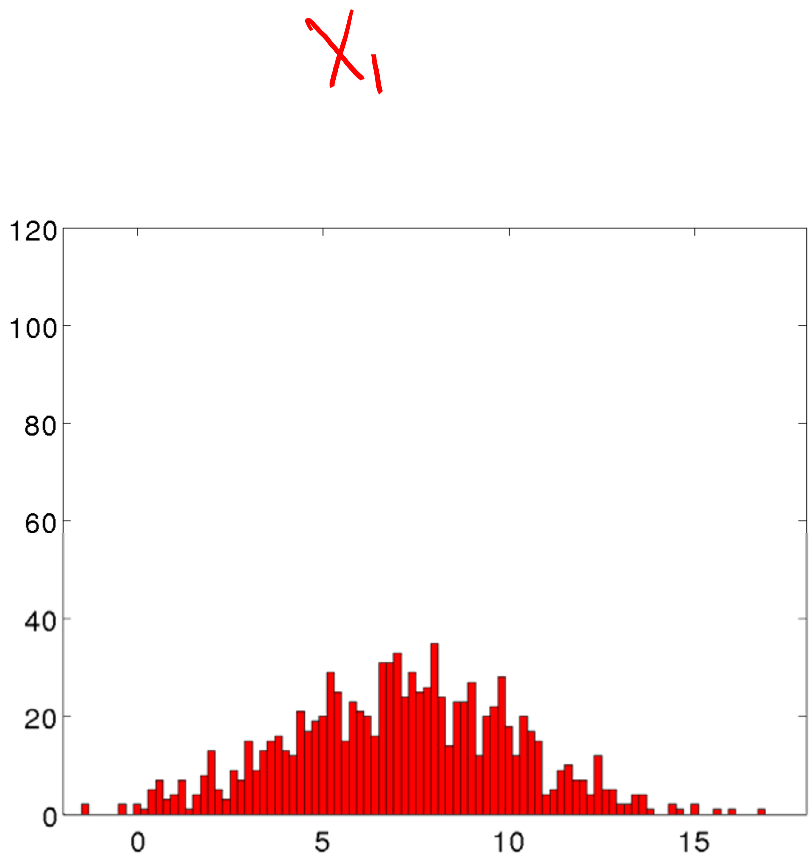
where ε_{σ^2} is zero-mean Gaussian noise with variance σ^2

Combining Noisy Measurements

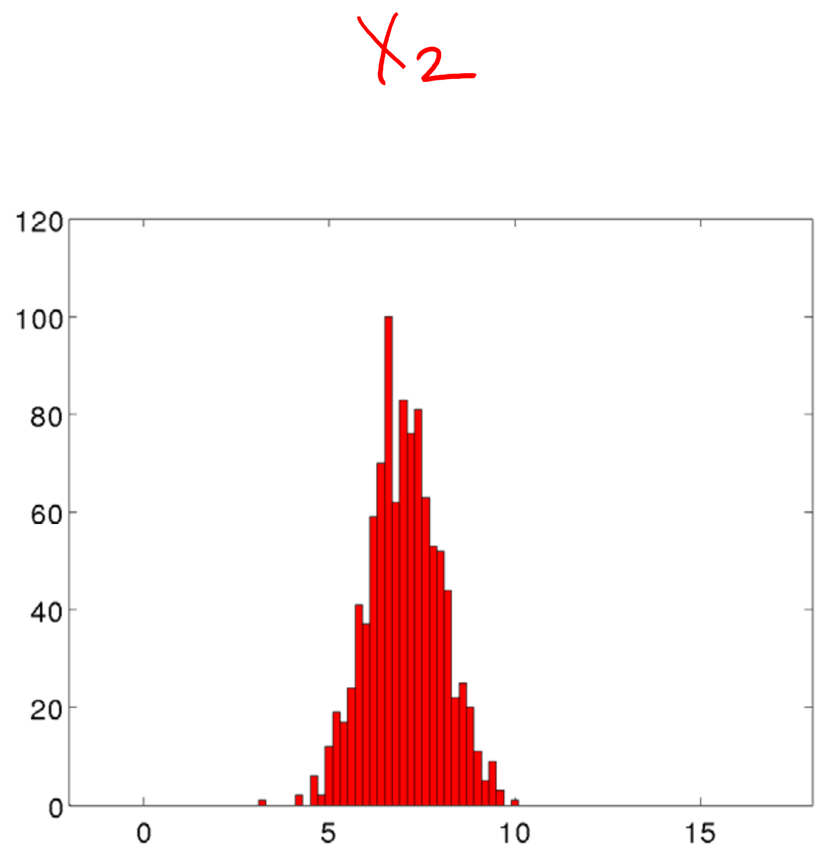
```
x = 7;
x1 = x + 3 * randn(1, 1000);   % noise variance = 3*3 = 9
x2 = x + randn(1, 1000);       % noise variance = 1
mu2 = (x1 + x2) / 2;

bins = -2:0.2:18;
hist(x1, bins);
hist(x2, bins);
hist(mu2, bins);
```

Combining Noisy Measurements



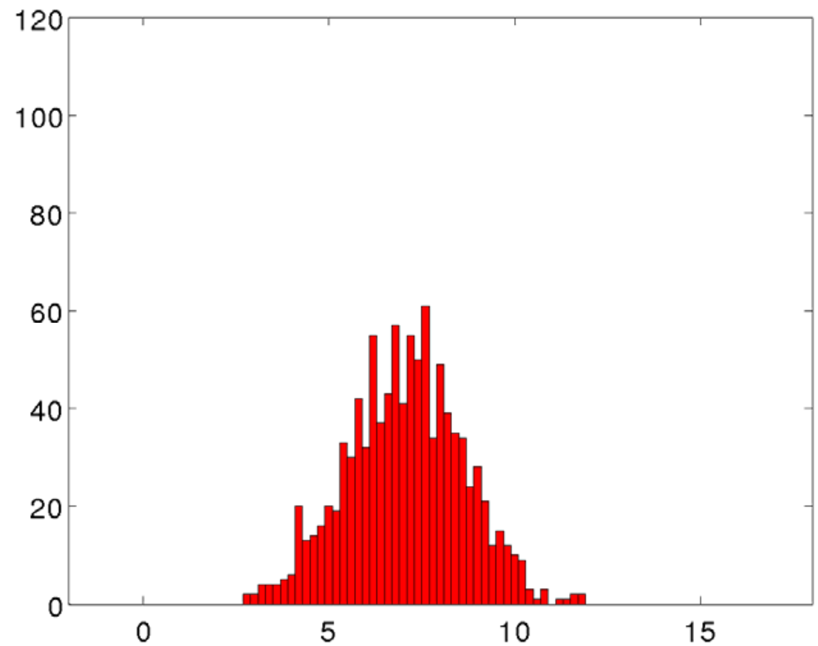
var(x1) = 8.9166



var(x2) = 0.9530

Combining Noisy Measurements

$$\mu = \frac{x_1 + x_2}{2}$$



var(mu2) = 2.4317

variance of average < var(x_1) but > var(x_2)

Combining Noisy Measurements

- is the average the optimal estimate of the combined measurements?

Combining Noisy Measurements

- instead of ordinary averaging, consider a weighted average

$$\mu = \omega_1 x_1 + \omega_2 x_2$$

  where $\omega_1 + \omega_2 = 1$
- the variance of a random variable is defined as

$$\mathrm{var}(X) = \mathrm{E}[(X - \mathrm{E}[X])^2]$$

  where $\mathrm{E}[X]$ is the expected value of X

Expected Value

- informally, the expected value of a random variable X is the long-run average observed value of X
- formally defined as

$$\mathrm{E}[X] = \int_{-\infty}^{\infty} x\, f(x)\, dx$$

- properties

$$\mathrm{E}[c] = c$$
$$\mathrm{E}[\mathrm{E}[X]] = \mathrm{E}[X]$$
$$\mathrm{E}[X + c] = \mathrm{E}[X] + c$$
$$\mathrm{E}[X + Y] = \mathrm{E}[X] + \mathrm{E}[Y]$$
$$\mathrm{E}[cX] = c\mathrm{E}[X]$$
$$\mathrm{E}[XY] = \mathrm{E}[X]E[Y] \text{ if } X \text{ and } Y \text{ are independent}$$

Variance of Weighted Average

$\mu = \omega_1 x_1 + \omega_2 x_2$

$$\begin{aligned}\mathrm{var}(\mu) &= \mathrm{E}[(\mu - \mathrm{E}[\mu])^2] \\ &= \mathrm{E}[(\omega_1 x_1 + \omega_2 x_2 - \mathrm{E}[\omega_1 x_1 + \omega_2 x_2])^2] \\ &= \ldots \\ &= \omega_1^2 \sigma_1^2 + \omega_2^2 \sigma_2^2 + 2\omega_1\omega_2 \mathrm{E}[(x_1 - \mathrm{E}[x_1])(x_2 - \mathrm{E}[x_2])]\end{aligned}$$

$= \mathrm{E}[x_1 x_2] - \mathrm{E}[x_1]\mathrm{E}[x_2]$

$= 0$ if x_1 and x_2 are independent

Variance of Weighted Average

- because x_1 and x_2 are independent

depends on x_1 depends on x_2

$$(x_1 - \mathrm{E}[x_1]) \quad \text{and} \quad (x_2 - \mathrm{E}[x_2])$$

are also independent; thus

$$\mathrm{E}[(x_1 - \mathrm{E}[x_1])(x_2 - \mathrm{E}[x_2])] = 0$$

$E[x_1 x_2] - E[x_1]E[x_2]$

$= E[x_1]E[x_2] - E[x_1]E[x_2]$

$= 0$

- finally

$$\mathrm{var}(\mu) = \omega_1^2 \sigma_1^2 + \omega_2^2 \sigma_2^2$$

Variance of Weighted Average

- because x_1 and x_2 are independent

$$(x_1 - \mathrm{E}[x_1]) \quad \text{and} \quad (x_2 - \mathrm{E}[x_2])$$

  are also independent; thus

$$\mathrm{E}[(x_1 - \mathrm{E}[x_1])(x_2 - \mathrm{E}[x_2])] = 0$$

- finally

$$\begin{aligned} \mathrm{var}(\mu) &= \omega_1^2 \sigma_1^2 + \omega_2^2 \sigma_2^2 \\ &= (1-\omega^2)\sigma_1^2 + \omega^2 \sigma_2^2 \quad \text{where } \omega_2 = \omega, \quad \omega_1 = 1 - \omega \end{aligned}$$

Variance of Weighted Average

- one way to choose the weighting values is to choose the weights such that the variance is minimized

$$\frac{d}{d\omega}\text{var}(\mu) = 0 = -2(1-\omega)\sigma_1^2 + 2\omega\sigma_2^2$$

$$\Rightarrow \omega = \frac{\sigma_1^2}{\sigma_1^2 + \sigma_2^2}$$

Minimum Variance Estimate

- thus, the minimum variance estimate is

$$\mu = \frac{\sigma_2^2}{\sigma_1^2 + \sigma_2^2} x_1 + \frac{\sigma_1^2}{\sigma_1^2 + \sigma_2^2} x_2$$

$$\mathrm{var}(\mu) = \frac{\sigma_1^2 \sigma_2^2}{\sigma_1^2 + \sigma_2^2}$$

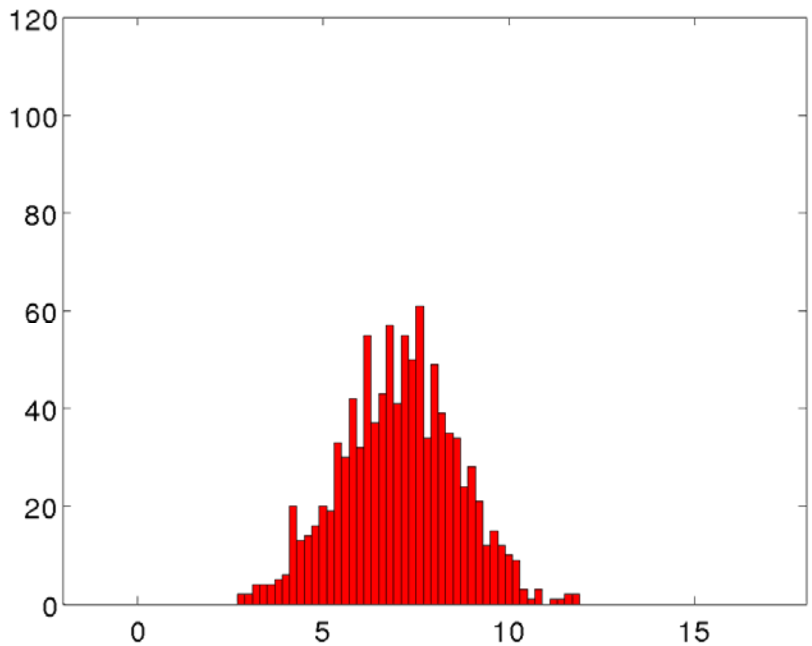
Combining Noisy Measurements

```
x = 7;
x1 = x + 3 * randn(1, 1000);   % noise variance = 3*3 = 9
x2 = x + randn(1, 1000);       % noise variance = 1
w = 9 / (9 + 1);
mu2 = (1 – w) * x1 + w * x2;

bins = -2:0.2:18;
hist(x1, bins);
hist(x2, bins);
hist(mu2, bins);
```

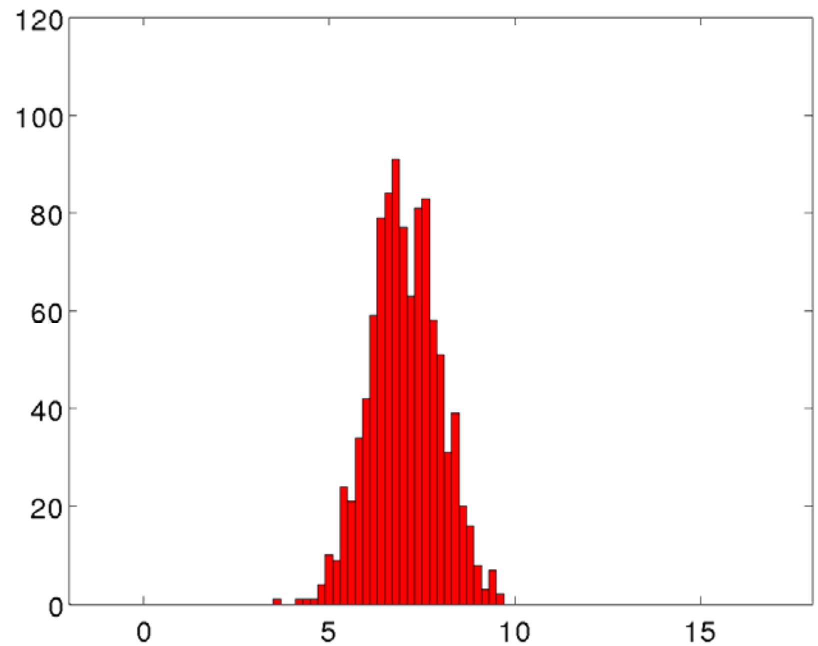
Minimum Variance Estimate

mu2=0.5*x1 + 0.5*x2



var(mu2) = 2.4317

mu2=0.1*x1 + 0.9*x2



var(mu2) = 0.8925 $< Var(x_1)$

$< Var(x_2)$